A Mathematical Basis for Wind Resources Prospecting

By

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Introduction

The goal of wind resource prospecting is to identify the locations and the energy resources of promising wind turbine sites. A more complete analysis of wind resources outside the scope of this report will include evaluations of ease of supply and economic cost effectiveness.

To most efficiently and accurately evaluate the energy content of wind resources it is essential to use a mathematical terminology that can relate to both turbine design and wind resource evaluation. Although much work has already been done in wind resource prospecting there is still no accepted common mathematical reference. The purpose of this section is to present a mathematical basis for wind prospecting.

Basic Energy and Power Calculations

A wind turbine generator may be envisioned as in Figure 1. E_{IN} is the total energy (watt-hr) in the wind incident on the turbine during a specified time interval. Work is the amount of E_{IN} that is applied to turning the turbine blades, E_{OUT} is the energy remaining in the wind after leaving the turbine, and E_{LOSS} is the energy lost to heat and turbulence in the conversion process.

The amount of work extracted divided by the incident energy is the coefficient of performance, C_p, of the turbine:

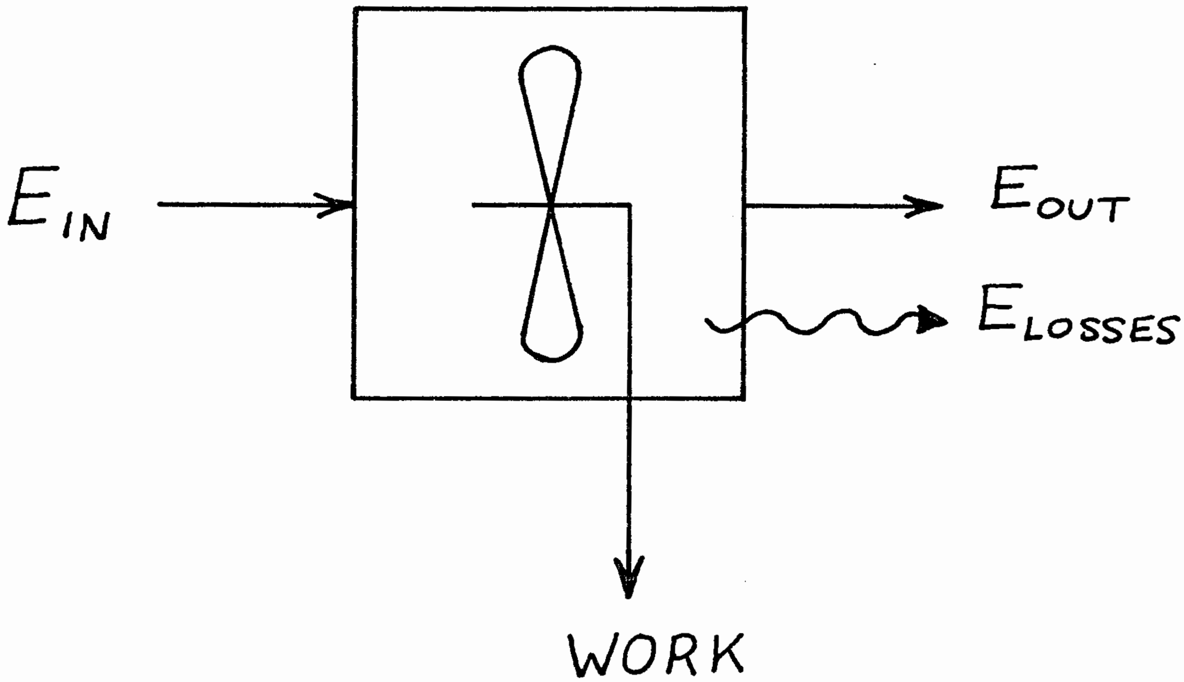


Fig. 1. Schematic of a wind turbine system, showing the energy in the wind entering the system, the energy in the wind leaving the system, the energy losses, and the work done on the turbine. By conservation, $E_{IN} = E_{OUT} + E_{LOSSES} + WORK$.

$$C_p = \frac{\text{Work}}{E_{IN}} \lesssim .593, \tag{1}$$

where the theoretical maximum value of C_p for a turbine efficiency of 100% is .593. This is the limit to the fraction of mechanical energy that can be extracted from the wind by a turbine. For turbine efficiencies, η, less than 100%, C_p is

$$C_p = .593\eta . \tag{2}$$

It is well known that the energy content, ΔE_{IN}, in the wind is

$$\Delta E_{IN} = \tfrac{1}{2}\rho A v_{IN}^3 \Delta t , \tag{3}$$

where ρ is the density of the air, v_{IN} is the speed of the wind, A is the cross sectional area of the wind of interest, or of the turbine, and Δt is the time interval. When considering properties of the wind independently of the size of the turbine it is more convenient to use energy density rather than energy. The energy density, $\mathcal{E}$, is the energy divided by A, or

$$\Delta \mathcal{E}_{IN} = \Delta E_{IN} / A , \tag{4}$$

where $\mathcal{E}$ is in watt-hr/m^2.

It is also helpful to think in terms of the rate of energy density flow into the turbine. This is found by allowing Δt to be small so that variations in time of v_{IN} are insignificant. The result is that the ratio of $\Delta\mathcal{E}_{IN}$ to Δt becomes equal to the instaneous rate of energy density flow. This is also known as the energy flux density or, simply the power density, $\partial\mathcal{E}_{IN} / \partial t$ (in watts/m^2), where

$$\frac{\partial \mathcal{E}_{IN}}{\partial t} = \frac{1}{2} \rho \, v_{IN}^{3} \, . \tag{5}$$

This is a quantity that will, in general, vary throughout the cross section of the turbine.

This is the information desired from wind prospecting but it is not what is generally practical to measure. Typical field anemometers measure v_{10}, the wind speed at 10 m above the surface. This gives

$$\frac{\partial \mathcal{E}_{10}}{\partial t} = \frac{1}{2} \rho \, v_{10}^{3} \, . \tag{6}$$

This information at 10 meters can be related to the information at the turbine through three approximations, as follows. First, we assume a hub height for the turbine; second, we assume that the wind, v_H, at hub height is related to the wind at 10 meters by a relationship giving

$$v_H^{3} = k_1 \, v_{10}^{3} \tag{7}$$

Third, we assume that V_H^3 will be closely related to V_{IN}^3 over the turbine area (to calculate power input from power density), or,

$$k_2 \, V_H^{3} = \frac{1}{A} \int V_{IN}^{3} \, dA \, , \tag{8}$$

where k_2 is approximately equal to one. Then the mean power input over the turbine area is

$$\frac{\partial \bar{\mathcal{E}}_{IN}}{\partial t} = \frac{1}{2} \rho \; k_1 \, k_2 \, V_{10}^{3} \, , \tag{9a}$$

$$= k_1 \, k_2 \, \frac{\partial \mathcal{E}_{10}}{\partial t} \, , \tag{9b}$$

where $\bar{\mathcal{E}}_{IN}$ is the mean energy density over the turbine area.

We have showed that the basic measured property in wind prospecting is $\partial \mathcal{E}_{10}/\partial t$ and how this can be related to the $\partial \bar{\mathcal{E}}_{IN}/\partial t$ for a prospective wind turbine. Following sections will deal with how to use $\partial \mathcal{E}_{10}/\partial t$ for wind resource evaluations and how to measure or approximate $\partial \mathcal{E}_{10}/\partial t$.

Resource Evaluation

After evaluating the wind power density at specific locations there remains a need to translate this wind resource information into a statement of area-wide energy potential. This area-wide evaluation may be performed by first finding the total energy density input at a possible turbine site for a year, or

$$\bar{\mathcal{E}}_{IN} = \int_T \frac{\partial \bar{\mathcal{E}}_{IN}}{\partial t}\, dt \,, \tag{10}$$

where the integral T is taken over a year. The dimensions of $\bar{\mathcal{E}}_{IN}$ are watt-hr/m^2 of turbine area. Then $\bar{\mathcal{E}}_{IN}$ is multiplied by k_3, a value representing the useable ratio of turbine area to land surface area, to get the available energy density, $\mathcal{E}_S$, over the land surface area, or

$$\mathcal{E}_S = k_3\, \bar{\mathcal{E}}_{IN} \tag{11}$$

$\mathcal{E}_S$ is in watt-hr/m^2 of land area. The ratio k_3 is dependent upon location. For flat terrain k_3 is approximately

$$k_3 = \frac{A}{S} = (110)^{-1}, \tag{12}$$

which is evaluated by assuming a hexagonal arrangement of wind turbines each separated from its nearest neighbor by 10 blade diameters. For mountain ridges over the land surface.

The total energy, E_S, over a specified surface area may be found by integrating the energy density $\mathcal{E}_S$ over the land surface,

$$E_S = \int_S \mathcal{E}_S \, dS , \tag{13}$$

where E_S is in watt-hr. Of course, not all of this energy is extractable. The extractable energy E_X is found by multiplying E_S by the turbines' coefficient of performance,

$$E_X = C_p E_S . \tag{14}$$

Example

Question

Estimated $\bar{\mathcal{E}}_{IN}$ = 600 T watt-hr/m^2 over a land surface of 25 mi^2, where T = 8760, the number of hours in a year. What is the energy available? What is the energy extractable? What is the average power extractable, $\bar{P}_X = E_X/T$? Let $k_3 = 100^{-1}$ and $C_p = 0.3$.

Answer

$$\mathcal{E}_S = k_3 \bar{\mathcal{E}}_{IN} = 6T \quad (\text{watt-hr/m}^2)$$
$$= 15.6\ T \quad (\text{Mw-hr/mi}^2)$$
$$E_S = \int \mathcal{E}_S \, dS = \mathcal{E}_S \cdot S = 400T \quad (\text{Mw-hr})$$
$$E_X = C_p E_S = 120T \quad (\text{Mw-hr})$$
$$P_X = E_X / T = 120 \quad \text{Mw}$$

Mathematical Representations of the Wind

Wind data can be displayed in many ways. Each way has its particular advantages and disadvantages. Our concern here is in selecting the most advantageous display for wind resource use.

Perhaps the most basic output of an anemometer is a record of wind speed versus time (see Figure 2), a display which has little direct use in wind prospecting because it does not convey a summary of wind behavior or a probability statement about its expected behavior. This wind speed record can be converted to a useful wind summary by breaking it up into small increments of time, finding the mean speed in each time increment, and adding the time increment to a bin that represents the mean speed in the increment. The result, after many entries, will be a histogram of the wind speed spectrum, as shown in Figure 3 (example from Schorran, 1978).

The goal of constructing this histogram should be to provide a good estimation of the corresponding density function which results from taking finer and finer bins in wind speed. The resulting density function is the time density function $t\langle v \rangle$ of wind speed illustrated in Figure 4a. It tells the amount of time the wind speed was found to be in the vicinity of the speed v. The total area under the curve is representative of the total hours of data processed. A more detailed discussion of density functions is given in Appendix A.

A second density function, which will turn out to be more useful than the time density function, is the energy density function $e\langle v \rangle$.

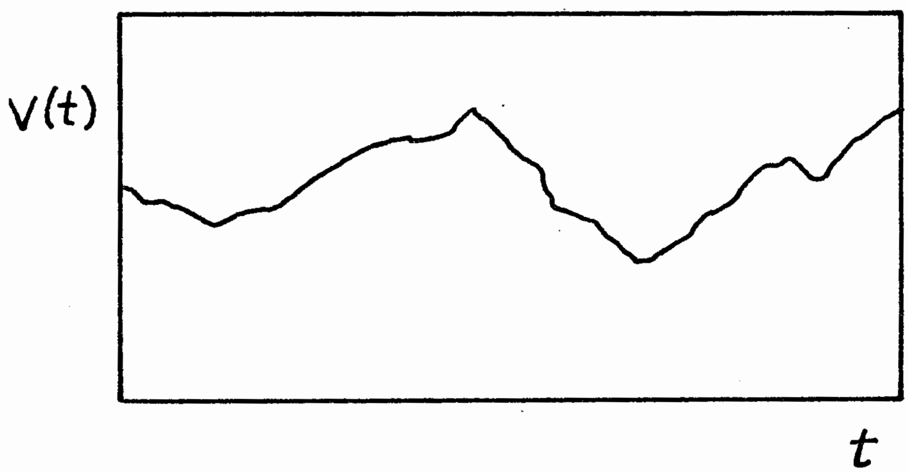


Fig. 2. Basic wind speed record produced by an anemometer.

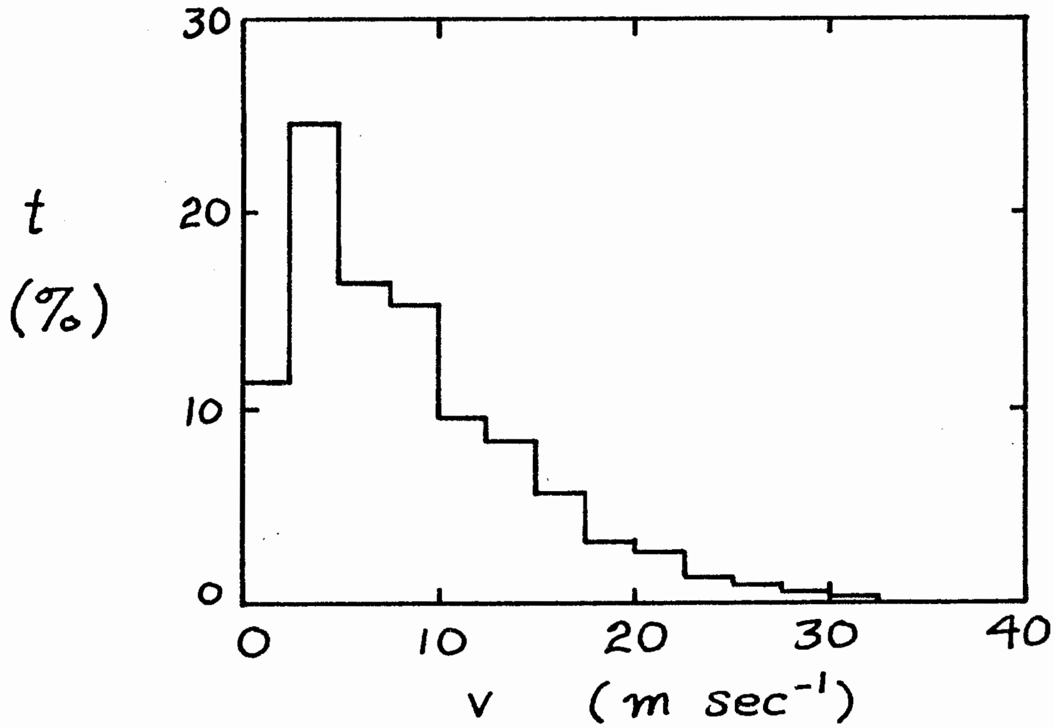


Fig. 3. Histogram of 1968 Slide Mountain 10 minute wind speed data calculated by Schorran (1978).

The energy density function is easily constructed from $t\langle v\rangle$ by using

$$e\langle v\rangle = \frac{1}{2}\rho v^3 \, t\langle v\rangle . \tag{15}$$

This function expresses the amount of energy found in the vicinity of wind speed v during the time of data recording. An example of $e\langle v\rangle$ is shown in Figure 4b. The total area under the curve is proportional to the total energy recorded. This is the most basic density function for wind resources needs. It shows the distribution of energy content over wind speed. It can be generalized to include direction as will be shown later.

After the wind speed record has been transformed from v(t) to $t\langle v\rangle$ it is well to note how the calculation of $\mathcal{E}_{10}$ or $\bar{\mathcal{E}}_{W}$ may be performed. From (6) we have

$$\mathcal{E}_{10} = \frac{1}{2}\rho \int_0^T v_{10}^3 \, dt . \tag{16}$$

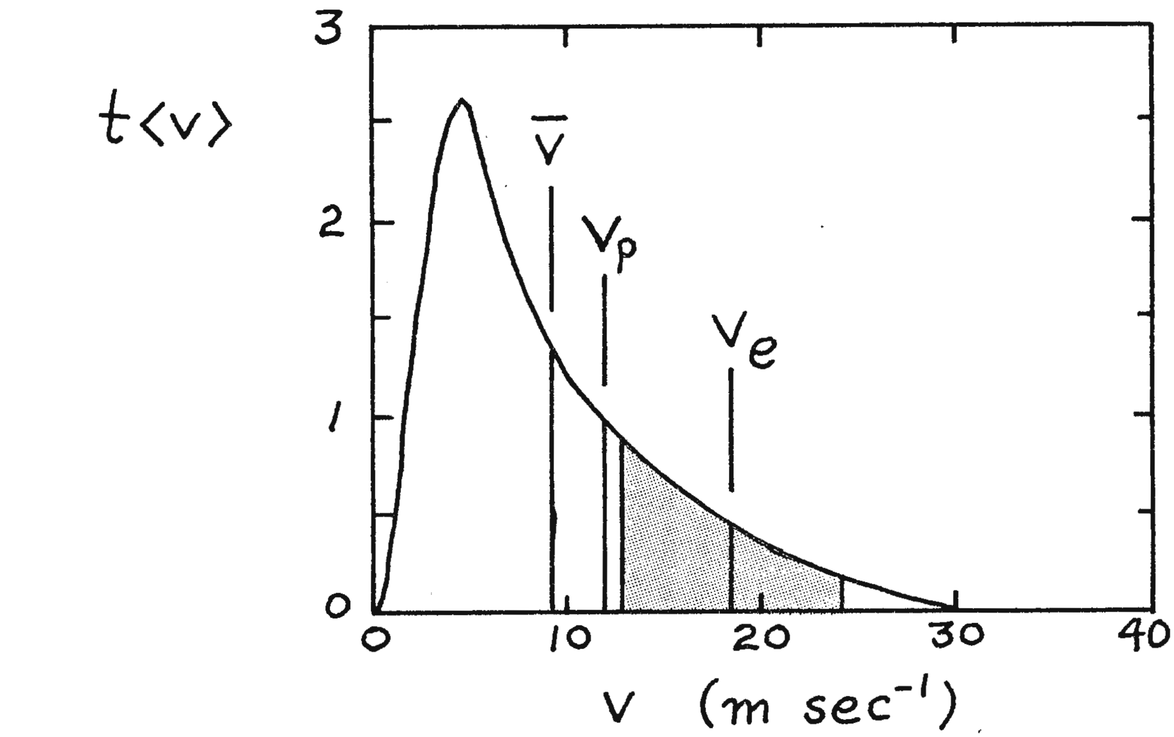
Using

$$T = \int_0^T dt = \int_0^\infty \frac{\partial T(v)}{\partial v} dv = \int_0^\infty t\langle v\rangle \, dv , \tag{17}$$

we have

$$\mathcal{E}_{10} = \frac{1}{2}\rho \int_0^\infty v_{10}^3 \, t\langle v\rangle \, dv . \tag{18}$$

The record of v (t) over time has been mapped (as illustrated in Figure 5) into the density function $t\langle v\rangle$ over velocity giving the



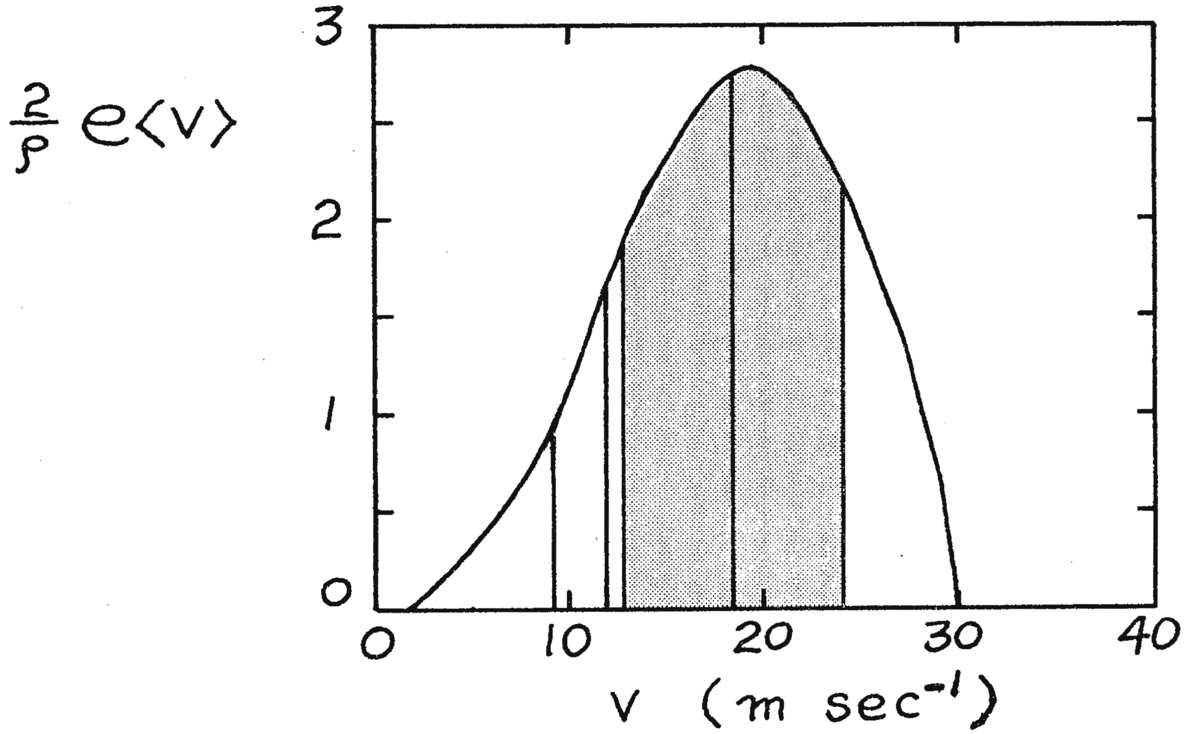


Fig. 4a. The time density function developed by smoothing the histogram data shown in Fig. 3. The spectrum moment $\bar{v}$ is the mean wind speed, v_p is the power equivalent speed, and v_e is the mean speed of the energy density function of Fig. 4b. Shaded area is the $1\sigma_e$ zone around v_e.

Fig. 4b. The energy density function calculated from Fig. 4a as described in the text.

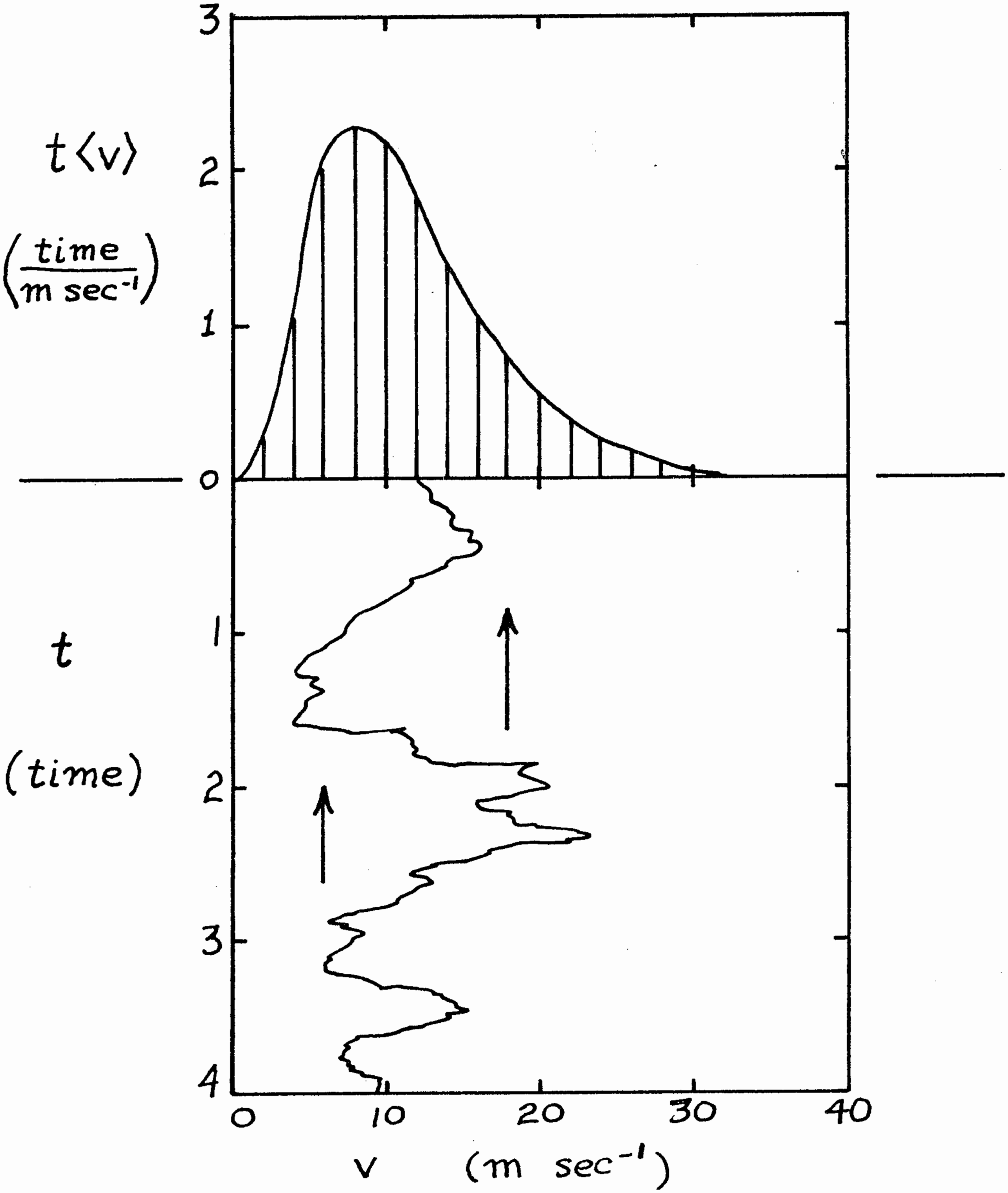


Fig. 5. Illustration of the mapping of an anemomter time record v(t) (as in Fig. 2), shown in the lower portion of the figure, into the time density function t⟨v⟩ , shown in the upper portion of the figure.

equivalent integrals of (17) where T(v) is the cumulative distribution function defined by

$$T(v) = \int_0^v t\langle v\rangle \, dv . \tag{19}$$

Inserting (15) in (18) we have

$$\mathcal{E}_{10} = \int_0^\infty e\langle v\rangle \, dv . \tag{20}$$

The energy density function $e\langle v\rangle$ has several uses. First, it is the only representation of wind data that displays the energy content of the different wind velocities over a period of time. If curve fitting it to be done then $e\langle v\rangle$ is the function that should be well matched throughout the area of higher $e\langle v\rangle$. Second, it shows that the important wind spectrum parameters are the mean v_e and standard deviation σ_e. These along with the integral $\mathcal{E}_{10}$ are the primary quantities relevant to wind resource evaluations, to turbine siting, and to turbine design. These are the desirable quantities to use for correlating wind records from different sites. These are the quantities that should be predicted or estimated when mean wind speeds are used as a basis for estimating wind energy. Let us illustrate these statements with a few examples.

Superimposed on Figure 4b are the mean speed v_e of $e\langle v\rangle$ and the range $(v_e - \sigma_e)$ to $(v_e + \sigma_e)$ shown shaded. These same values are reflected on the $t\langle v\rangle$ plot of Figure 4a. It is seen that v_e and the shaded area are truly representative of the energy content of the wind spectrum.

Also shown in Figures 4a and b is the mean speed $\overline{V}$ of the time density plot $t\langle v\rangle$. This is the mean speed usually used in wind analysis, and is the speed resulting from "wind run" anemometers. It is seen that $\overline{V}$ is not representative of the important part of the energy spectrum. The important question, of course, is, How well can the key parameters of the wind energy spectrum be predicted from $\overline{V}$?

Another parameter sometimes used in wind resource analysis is the "power equivalent speed," V_p, defined from (6), as the speed which when inserted in (6) will give mean power, or

$$\frac{\partial \mathcal{E}_{10}}{\partial t} = \frac{1}{2} \rho V_p^3 . \tag{21}$$

From (16) and (18) we find this is equivalent to

$$V_p^3 = \frac{1}{T} \int_0^T V_{10}^3 \, dt , \tag{22}$$

and

$$V_p^3 = \frac{1}{T} \int_0^\infty V_{10}^3 \, t\langle v\rangle \, dv. \tag{23}$$

The power equivalent speed, V_p, is also shown in Figure 4a and b. It is seen that V_p is also not representative of the important part of the wind power spectrum. It does not even come within one σ_e of V_e.

In summary, the fundamental properties of wind information for purposes of wind resources evaluations are the energy density function $e\langle v\rangle$ its mean V_e and standard deviation σ_e and its total integral $\mathcal{E}$. The importance of these values will be further illustrated in the following section.

Wind Analysis Calculations

The wind energy density function $e\langle v\rangle$ and its several parameters are derived from the time density function $t\langle v\rangle$. Also, the mean wind speed $\bar{v}$ and the power equivalent speed v_p may be derived from $t\langle v\rangle$. This is best illustrated by defining five integrals of $t\langle v\rangle$ as follows:

$$T = \int t\langle v\rangle\, dv \,, \tag{24}$$

$$D = \int v\, t\langle v\rangle\, dv \,, \tag{25}$$

$$\mathcal{E}_0 = \frac{\rho}{2}\int v^3\, t\langle v\rangle\, dv = \int e\langle v\rangle\, dv \,, \tag{26}$$

$$\mathcal{E}_1 = \frac{\rho}{2}\int v^4\, t\langle v\rangle\, dv = \int v\, e\langle v\rangle\, dv \,, \tag{27}$$

$$\mathcal{E}_2 = \frac{\rho}{2}\int v^5\, t\langle v\rangle\, dv = \int v^2\, e\langle v\rangle\, dv \,, \tag{28}$$

where T is the measurement time, D is the distance traveled by the wind (as might be recorded on an odometer), $\mathcal{E}_0$ is the total energy in the recorded wind, and $\mathcal{E}_1$ and $\mathcal{E}_2$ are the first and second moments integrals of $e\langle v\rangle$.

Using these integrals the following four speeds may be calculated:

$$\bar{v} = D/T \tag{29}$$

$$v_p = \left(\frac{2}{\rho}\mathcal{E}_0/T\right)^{1/3} \tag{30}$$

$$v_e = \mathcal{E}_1/\mathcal{E}_0 \tag{31}$$

$$v_f = \left(\mathcal{E}_2/\mathcal{E}_0\right)^{1/2} \tag{32}$$

where $\bar{v}$ is the mean speed and v_p is the power equivalent speed. We see that the reason that v_p is not representative of the energy spectrum is that its base is the integral T of the time density function and not the integral $\mathcal{E}_o$ of the energy density function.

The standard deviation σ_e of e⟨v⟩ is found as follows:

$$\begin{aligned}
\sigma_e{}^2 &= \frac{1}{\mathcal{E}_o} \int (v - v_e)^2 \, e\langle v\rangle \, dv \\
&= \frac{1}{\mathcal{E}_o} \int (v^2 - 2 v v_e + v_e^2) \, e\langle v\rangle \, dv \\
&= \frac{\mathcal{E}_2}{\mathcal{E}_o} - 2 \frac{\mathcal{E}_1}{\mathcal{E}_o} v_e + v_e^2 \\
&= v_f{}^2 - v_e^2
\end{aligned}$$

or

$$\sigma_e = (v_f^2 - v_e^2)^{1/2} \tag{33}$$

This should not be confused with the standard deviation about $\bar{v}$ of the time density function.

We will not illustrate these calculations by using Figures 4a and 4b as examples.

Example

Question

Using Fig. 4a as the wind data base, find v, v_p, v_e, σ_e, $(2/\rho)\,\mathcal{E}_o$, and $(2/\rho)\, e\langle v\rangle$ at even values of v. Construct Figure 4b.

Solution

Construct a table as shown in Table 1. First list v then list list t⟨v⟩ found by reading Figure 4a as well as possible. (In

Table 1

v	$t\langle v\rangle$	$vt\langle v\rangle$	$v^3t\langle v\rangle$	$v^4t\langle v\rangle$	$v^5t\langle v\rangle$
0	.00	.00E1	.00E3	.00E4	.00E5
2	1.70	.34	.01	.00	.00
4	2.60	1.04	.17	.07	.00
6	2.00	1.20	.43	.26	.16
8	1.50	1.20	.77	.61	.49
10	1.18	1.18	1.18	1.18	1.18
12	.95	1.14	1.64	1.97	2.36
14	.75	1.05	2.06	2.88	4.04
16	.59	.94	2.42	3.87	6.20
18	.43	.77	2.51	4.52	8.13
20	.35	.70	2.80	5.60	11.20
22	.22	.48	2.34	5.15	11.33
24	.15	.36	2.07	4.97	11.92
26	.10	.26	1.76	4.58	11.90
28	.05	.14	1.10	3.08	8.62
30	.00	.00	.00	.00	.00
	2.52E1	2.34 E2	4.26 E4	7.74E5	15.52E6
	$= T$	$= D$	$=(2/\rho)\mathcal{E}_0$	$=(2/\rho)\mathcal{E}_1$	$=(2/\rho)\mathcal{E}_2$

In arbitrary units.

Note that the summed column's have been multiplied by 2 to account for the fact that we have used Δv = 2m/sec to approximate the integral:

$$T = \int t\langle v\rangle \, dv \doteq \sum t\langle v\rangle \, \Delta v = 2 \sum t\langle v\rangle .$$

practice, a table of $t\langle v\rangle$ would normally be provided directly from the data base.) From these values calculate $vt\langle v\rangle$, $v^3t\langle v\rangle$, $v^4t\langle v\rangle$, and $v^5t\langle v\rangle$ as shown in Table 1. Sum each column (except v) and multiply by 2 to account for using $\Delta v = 2$, and find T, D, $\mathcal{E}_o$, $\mathcal{E}_1$, and $\mathcal{E}_2$ in arbitrary units.

The column $v^3t\langle v\rangle$ is $(2/\rho)\, e\langle v\rangle$ and these values may be plotted to give Figure 4b. The sum at the bottom of this column is $(2/\rho)\, \mathcal{E}_o$. Using the sums in Table 1 and equations (29) through (33), we calculate

$$\bar{V} = 9.3 \quad m/sec$$
$$V_p = 11.9 \quad m/sec$$
$$V_e = 18.2 \quad m/sec$$
$$V_f = 19.1 \quad m/sec$$
$$\sigma_e = 5.8 \quad m/sec$$

$$V_e - \sigma_e = 12.4 \quad m/sec$$
$$V_e + \sigma_e = 24.0 \quad m/sec$$

Question

Assume that the data was taken for one year and that $\rho = 0.95$ kg/m^3. Find $\mathcal{E}_o$ in watt yr/m^2 and replot Figures 4a and 4b in units of yr(m/sec)$^{-1}$ and (watt yr/m^2)(m/sec)$^{-1}$, respectively.

Solution

Since it is given that T = 1 (yr) we may adjust the scales of Figs. 4a and 4b by dividing all entries (except v) in Table 1 by 25.2, the value of T in Table 1. Note that this re-scaling will not change the values of the v's and σ_e, since these were developed from ratios of the primary integrals.

After division by 25.2, $t\langle v\rangle$ may be plotted in proper dimensions as shown in Figure 6a. The new values of $e\langle v\rangle$ are found by multiplying by $\rho/2$ giving

$$e\langle v\rangle = (\rho/2)\ [v^3 t\langle v\rangle] \div 25.2$$
$$= 1.885E\text{-}2\ [v^3 t\langle v\rangle]\ (\text{watt yr}/m^2)(m/\text{sec})^{-1}$$

where $[v^3 t\langle v\rangle]$ refers to the values in column 4 of Table 1 and the dimensions have been found from $\frac{1}{2}\rho v^3$ as follows:

$$1\left(\frac{Kg}{m^3}\right)\left(\frac{m}{sec}\right)^3 = 1\left(\frac{\text{joule}}{m^2\ sec}\right) = 1\left(\frac{\text{watt}}{m^2}\right)$$

The new plot of $e\langle v\rangle$ is shown in Figure 6b.

Following the division by 25.2, we find $\mathcal{E}_o$ as follows

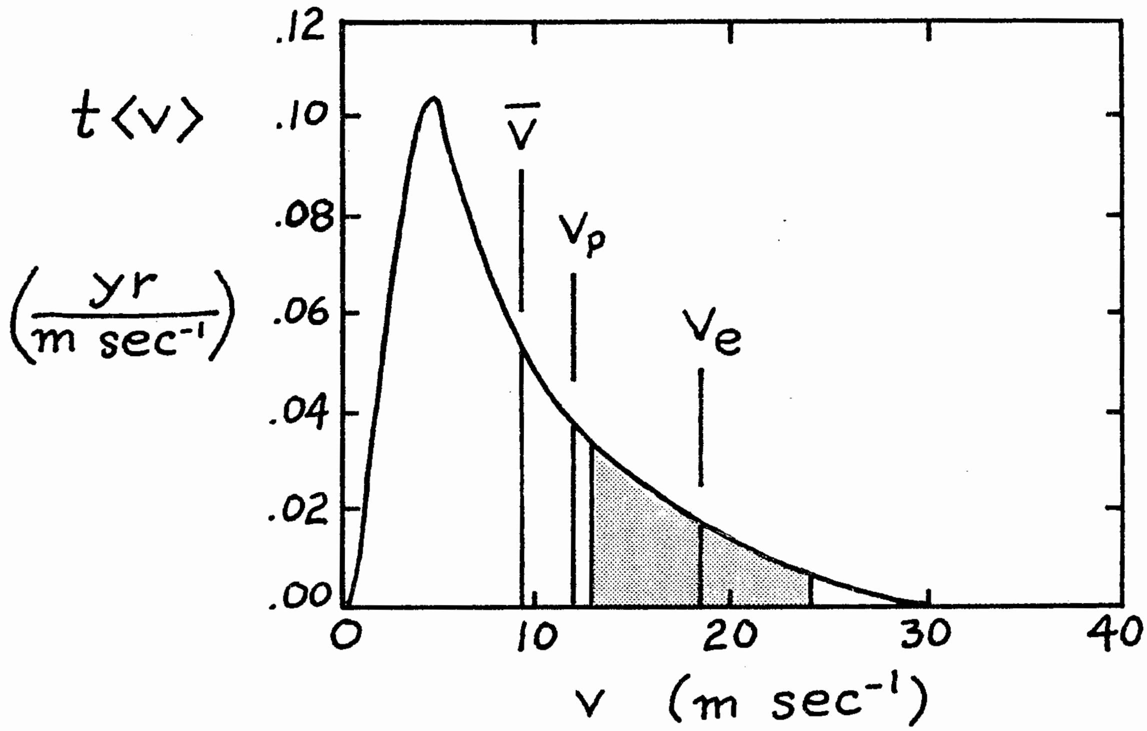
$$\mathcal{E}_o = \frac{\rho}{2}\ 1.69E+3$$
$$= 803\ (\text{watt yr}/m^2)$$

Question

Using the value of $\mathcal{E}_o$ calculated above calculate the mean during a year.

Solution

$$\bar{P} = \mathcal{E}_o / T$$
$$= 803\ (\text{watt}/m^2)$$



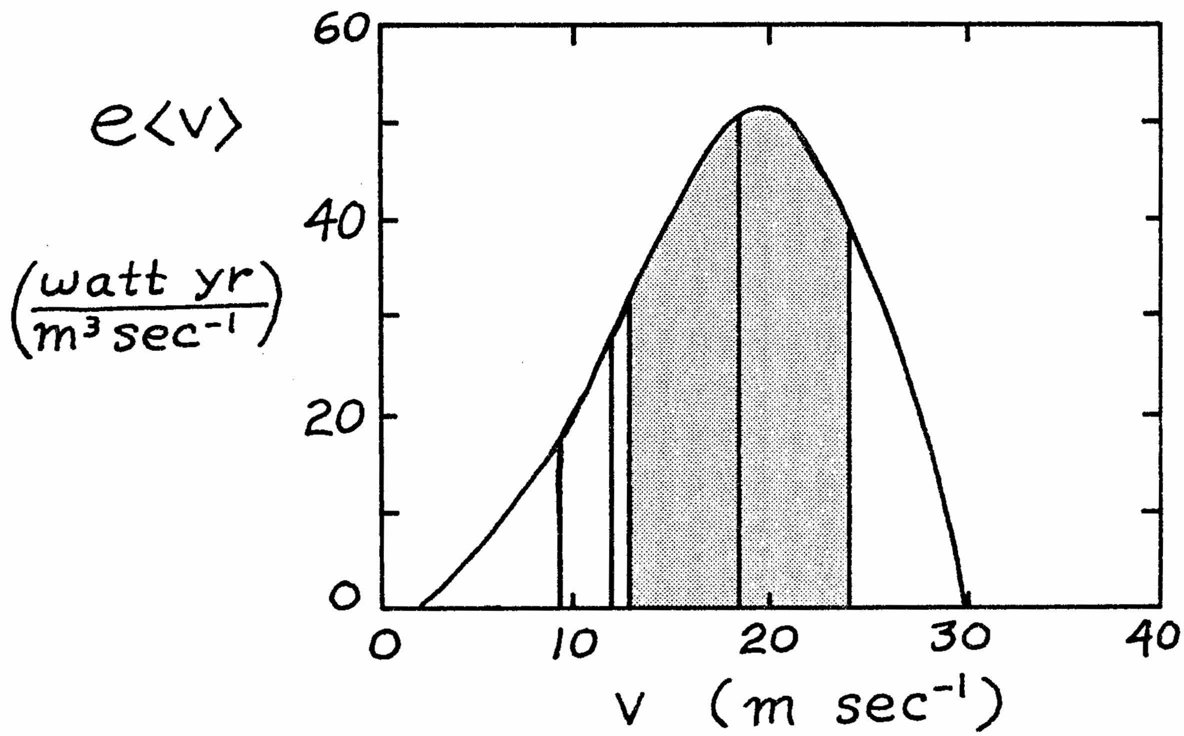
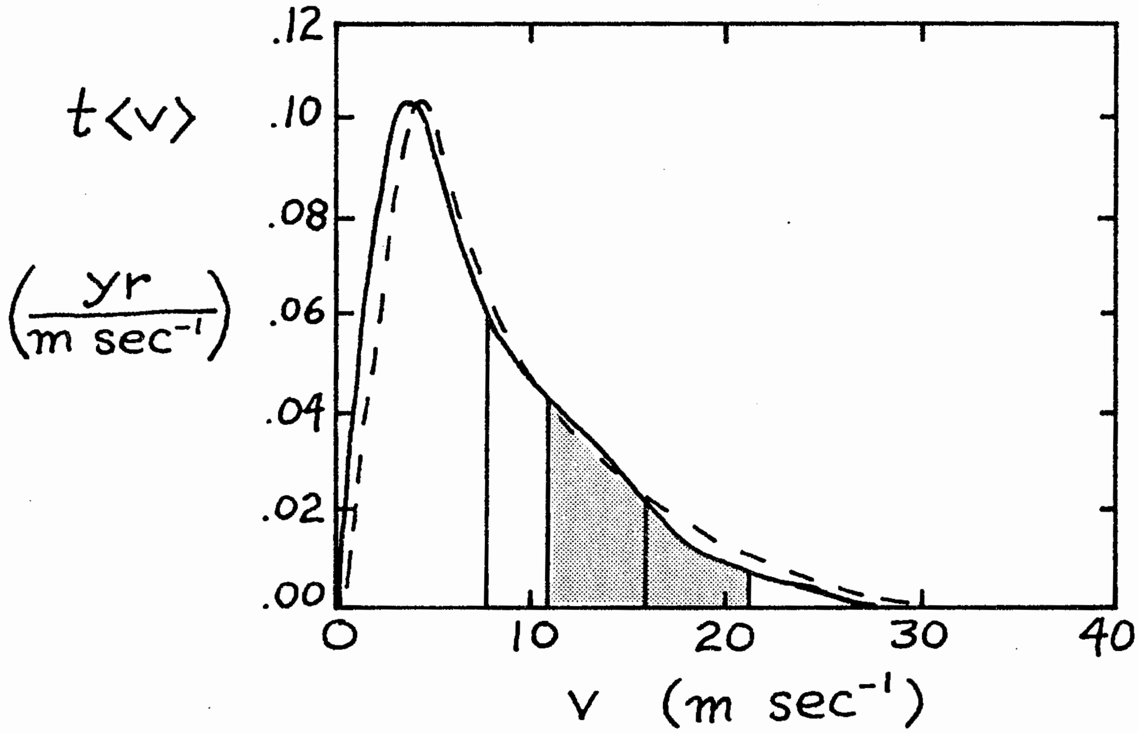


Fig. 6a. The time density function of Fig. 4a normalized as described in the text. Spectral moments and shading have the same meaning as in Fig. 4. Developed from histogram data calculated by Schorran (1978) for 1968 Slide Mountain 10 minute data.

Fig. 6b. The energy density function corresponding to Fig. 6a.

In summary, we see that the energy density function $e\langle v\rangle$ is central to wind energy considerations and is easily related to $t\langle v\rangle$. The key properties of the energy density function and the time density function can be expressed in terms of 5 easily obtained integrals. These integrals in turn can be used to calculate the key moments of a wind spectrum.



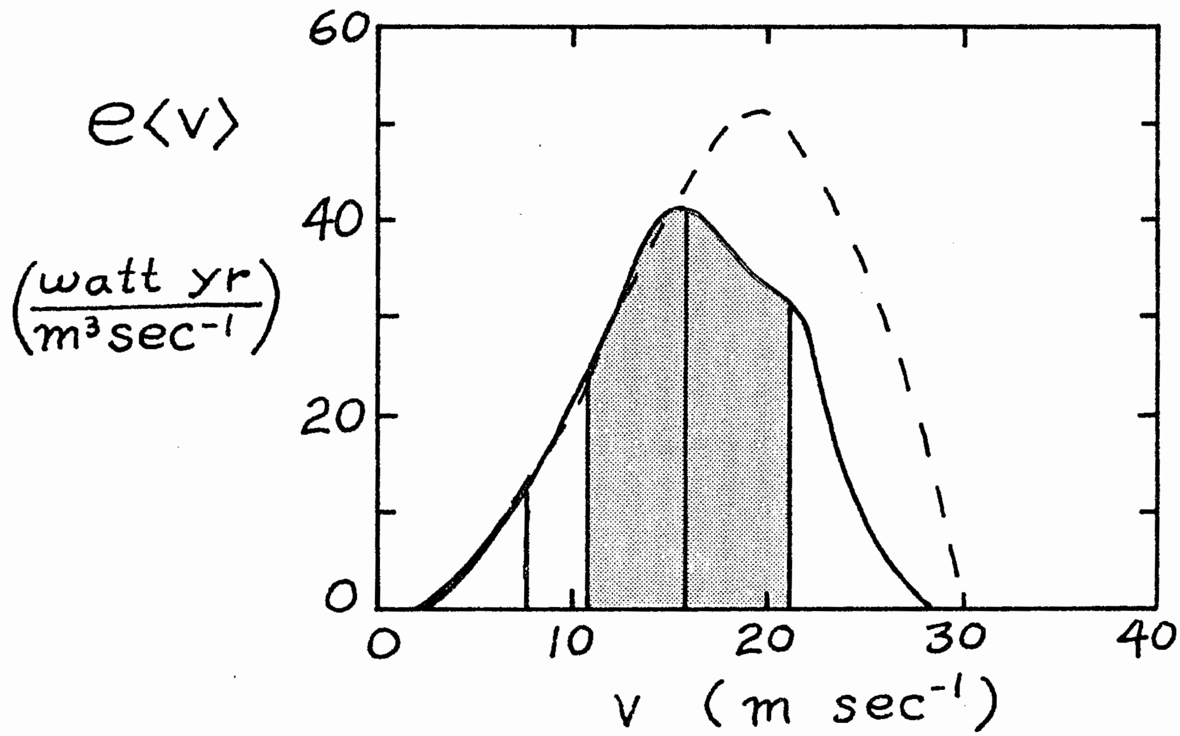
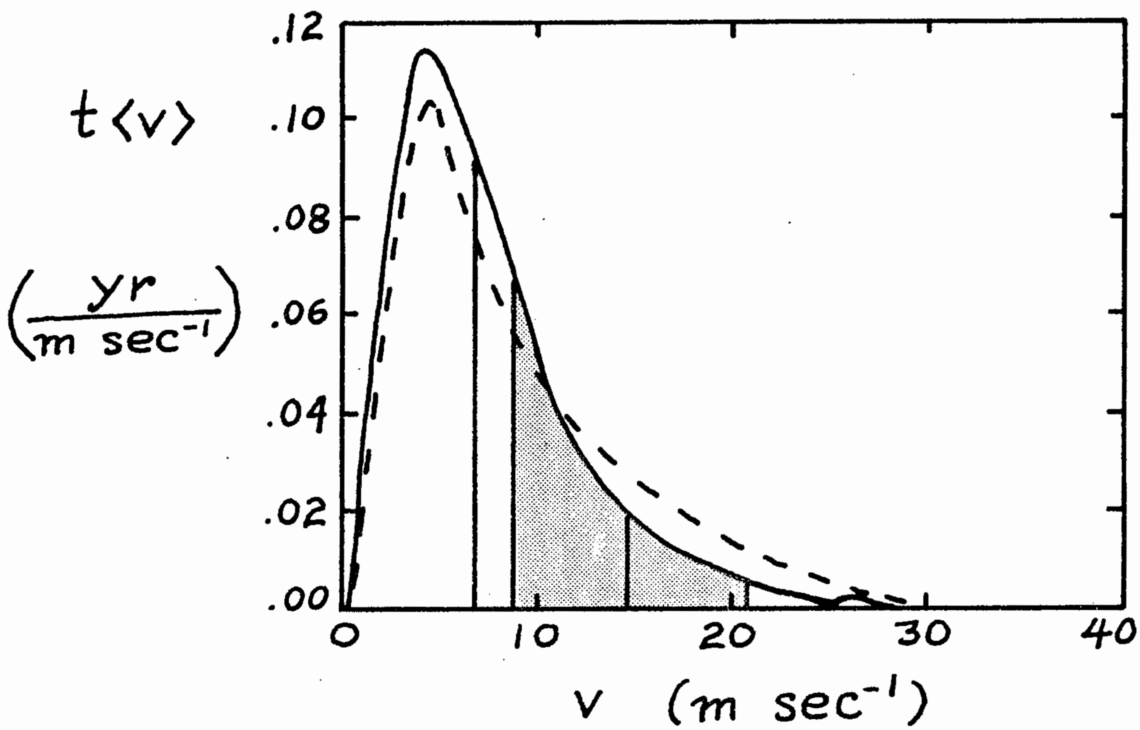


Fig. 7a. The time density function for 3 hr. average 1968 Slide Mountain data developed from histogram calculated by Schorran (1978).

Fig. 7b. The energy density function corresponding to Fig. 7a.



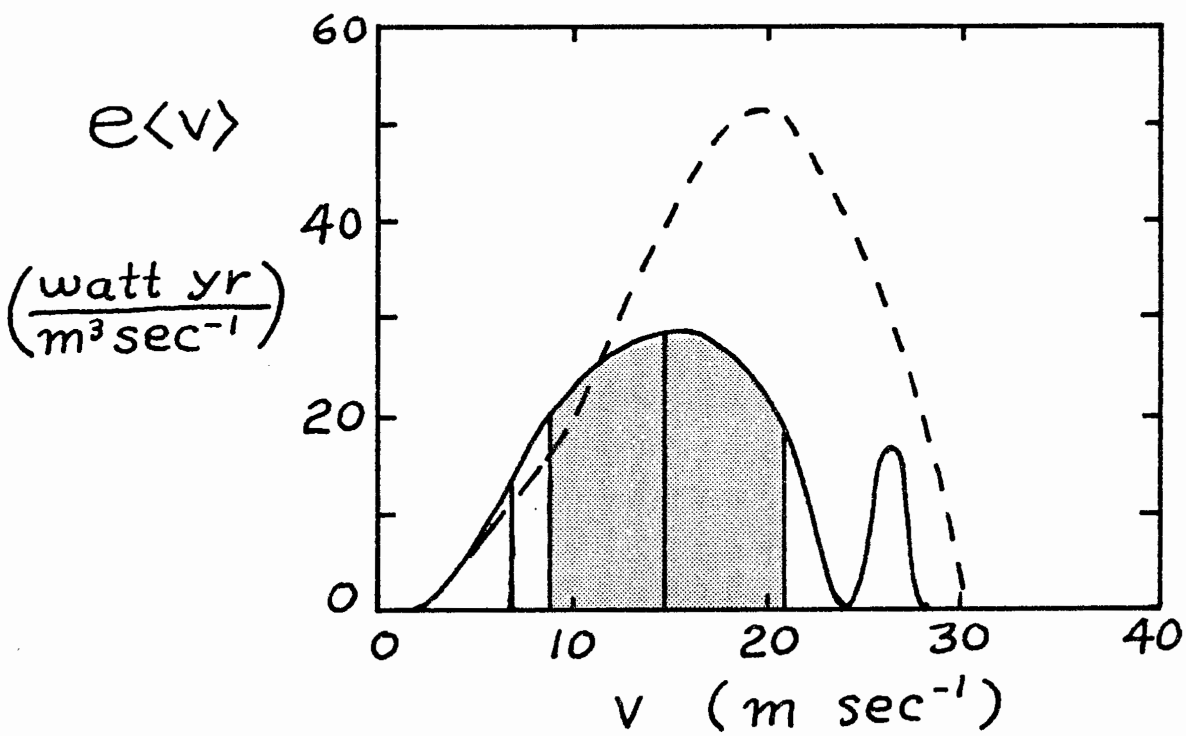


Fig. 8a. The time density function for 1968 Reno pibal data at approximate Slide Mountain elevation developed from histogram calculated by Schorran (1978).

Fig. 8b. The energy density function corresponding to Fig. 8a.

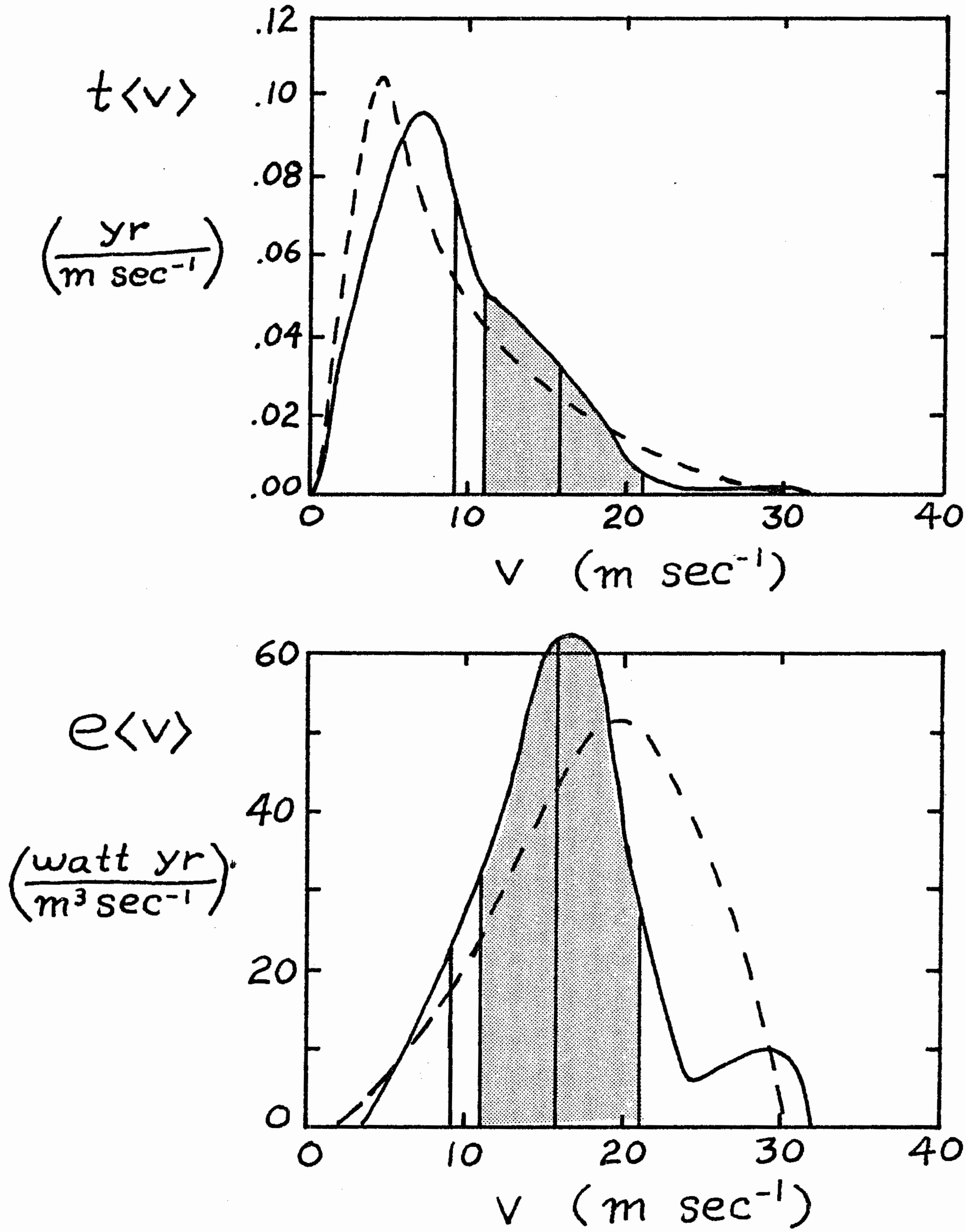


Fig. 9a. The time density function for 1968 geostrophic winds at 700 mb over Slide Mountain developed from histogram calculated by Schorran (1978).

Fig. 9b. The energy density function corresponding to Fig. 9a.

ENERGY FLUX DENSITY

$$\Delta E_{IN} = \frac{1}{2} \rho A v^3 \cdot \Delta t$$

$$\Delta \mathcal{E}_{IN} = \frac{1}{2} \rho v^3 \cdot \Delta t$$

$$\frac{\partial \mathcal{E}_{IN}}{\partial t} = \frac{1}{2} \rho v^3$$

ENERGY DENSITY FUNCTION

$$T = \int_0^T dt = \int_0^\infty t\langle v\rangle \, dv$$

$$\mathcal{E}_{IN} = \frac{\rho}{2} \int_0^T v^3 \, dt$$

$$= \frac{\rho}{2} \int_0^\infty v^3 \, t\langle v\rangle \, dv$$

$$= \int_0^\infty e\langle v\rangle \, dv$$

$$e\langle v\rangle = \frac{\rho}{2} v^3 \, t\langle v\rangle$$

FIVE BASIC INTEGRALS

$$T = \int t\langle v\rangle \, dv$$

$$D = \int v \, t\langle v\rangle \, dv$$

$$\mathcal{E}_0 = \frac{\rho}{2} \int v^3 \, t\langle v\rangle \, dv$$

$$\mathcal{E}_1 = \frac{\rho}{2} \int v^4 \, t\langle v\rangle \, dv$$

$$\mathcal{E}_2 = \frac{\rho}{2} \int v^5 \, t\langle v\rangle \, dv$$

FIVE INTENSIVE VARIABLES

$$\bar{V} = \frac{D}{T}$$

$$V_p = \left(\frac{\frac{2}{\rho} \mathcal{E}_0}{T} \right)^{1/3}$$

$$V_e = \frac{\mathcal{E}_1}{\mathcal{E}_0}$$

$$V_f = \left(\frac{\mathcal{E}_2}{\mathcal{E}_0} \right)^{1/2}$$

$$\sigma_e = \left(V_f^{\,2} - V_e^{\,2} \right)^{1/2}$$