Best Regards
WMB

I AM NOT AN AUTHOR BUT I DISCOVERED THIS MATH RELATIONSHIP IN A GRADUATE MATH CLASS TAUGHT BY PROF. BEESLEY.

HEY, HE GAVE ME AN "A".

EXB

Reprinted from the AMERICAN MATHEMATICAL MONTHLY
Vol. 76, No. 4, April, 1969

AN INTEGRAL REPRESENTATION FOR THE EULER NUMBERS

E. M. BEESLEY, University of Nevada, Reno

As is well known, [3], the Euler numbers satisfy the symbolic equation

$$(*) \qquad (E + 1)^n + (E - 1)^n = \begin{cases} 2 & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $E_0=1$, $E_1=0$, $E_2=-1$, $E_3=0$, $E_4=5$, $\cdots$. Although it can be deduced rather easily from a known integral ([1] p. 555, [5]), the fact that, for each nonnegative integer n,

$$E_n = (i)^{-n}\left(\frac{2}{\pi}\right)^{n+1}\int_0^\infty\left(\frac{(\log(x))^n}{1+x^2}\right)dx \tag{1}$$

seems not to be well known. It is evident that this formula gives the desired zero value whenever n is odd. For $n=0$, evaluation of the integral is a familiar calculus problem and, with $n=2$, it has appeared as a higher level problem [4], but the connection with the Euler numbers is hardly discernible from these special cases. In the interest of making this note self-contained, we give a derivation which is, in part, the same as that used by Glasser [2] in establishing a more general recursion formula.

If $r>0$, $0\leqq\theta\leqq\pi$ and $z=r\exp(i\theta)$, we let $\log z=\log r+i\theta$; and we let C represent the contour consisting of the real axis from r to R, the upper half of the circle $|z|=R$, the real axis from $-R$ to $-r$ and the upper half of the circle $|z|=r$. Then $\log i=i\pi/2$ and, if r is sufficiently small and R is sufficiently large, the Cauchy integral formula yields

$$\int_r^R \frac{(\log x)^n}{1+x^2}dx+\int_R^{-R}\frac{(\log z)^n}{1+z^2}dz+\int_{-R}^{-r}\frac{(\log(-x)+i\pi)^n}{1+x^2}dx+\int_{-r}^{r}\frac{(\log z)^n}{1+z^2}dz$$

$$=\int_C\frac{(\log z)^n}{z^2+1}dz = 2\pi i(i\pi/2)^n/2i.$$

Using a direct attack, one shows easily that the integrals on the semicircles go to zero as $r\to 0$ and $R\to\infty$. Hence, by using "$-x$ for x" in one integral and then the binomial theorem, we obtain

$$\int_0^\infty\frac{(\log x)^n}{1+x^2}dx+\sum_{k=0}^n\binom{n}{k}(i\pi)^{n-k}\int_c^\infty\frac{(\log x)^k}{1+x^2}dx=\pi\left(\frac{i\pi}{2}\right)^n.$$

If we multiply both sides by $i^{-n}(2/\pi)^{n+1}$, we have

$$(i)^{-n}\left(\frac{2}{\pi}\right)^{n+1}\int_0^\infty\frac{(\log x)^n}{1+x^2}dx + \sum_{k=0}^n\binom{n}{k}2^{n-k}(i)^{-k}\left(\frac{2}{\pi}\right)^{k+1}\int_0^\infty\frac{(\log x)^k}{1+x^2}dx = 2. \tag{2}$$

Upon setting $E_k'=(i)^{-k}(2/\pi)^{k+1}\int_0^\infty(\log x)^k dx/(1+x^2)$, equation (2) can be written symbolically

$$(E')^n+(E'+2)^n=2. \tag{3}$$

We now notice that the fundamental identity (*) generalizes at once to $f(E+1)+f(E-1)=2f(0)$ where f is any polynomial function. Taking $f(x)=(x+1)^n$ we have $(E+2)^n+(E)^n=2$.

By comparing this with (3) and noticing that $E_0'=1=E_0$ we see that $E_n=E_n'$ for each nonnegative n and that (1) is established.

References

1. D. Bierens de Haan, Memoirs of the Royal Academy of Sciences, vol. VIII, Amsterdam, 1862.

2. M. L. Glasser, Some recursive formulas for evaluation of a class of definite integrals, this Monthly, 72 (1964) 75–76.

3. K. Knopp, Theorie und Anwendung der unendlichen Reihen, Springer, Berlin, 1931.

4. E. C. Titchmarsh, The Theory of Functions, Oxford, London, 1932.

5. D. Zeitlin, On a class of definite integrals, this Monthly, 75 (1968) 878–879.