A Mathematical Framework for Cloud Models

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ABSTRACT

A mathematical framework is described which allows the logical joining of the processes of condensation, collection and advection, and also freezing and electrical effects in a cloud model.

1. Introduction

Cloud models developed by several investigators treat separately and sometimes jointly the processes of condensation, collection, advection, freezing and melting. Problems will usually arise when these processes are joined into a more comprehensive cloud model, especially if the separate parts have been developed by different individuals. The structure described here is *one* method by which we can "talk the same language." Problems in the composition of cloud models should thereby be reduced.

Since the purpose of this framework is to be general, the equations will be general, but this does not preclude the individual simplification of any of the structural parts. Also, one can reduce the droplet density function to two or three "classes" of cloud particles. The general structure remains unchanged.

Since the general equation is interesting in itself, we shall finish with a few digressions into its nature.

2. General description

We use the term *particle* to mean any type of hydrometeor and distinguish a hydrometeor by its mass, its type and its position. The type of hydrometeor is recorded by its *internal parameters* which can isolate its nucleus mass, its electric charge, its phase and crystal type, etc.

Rather than attempt to record each particle individually, we record the presence of particles of different size and internal parameters by a density function over particle size, internal parameters and spatial position.

To complete the formulation, we need to specify those parameters of the *environment* which affect the rates of change of mass, type or position of the particles. The particles in turn affect their local environment through the exchange of water mass, heat, etc. A set of interface equations must be written.

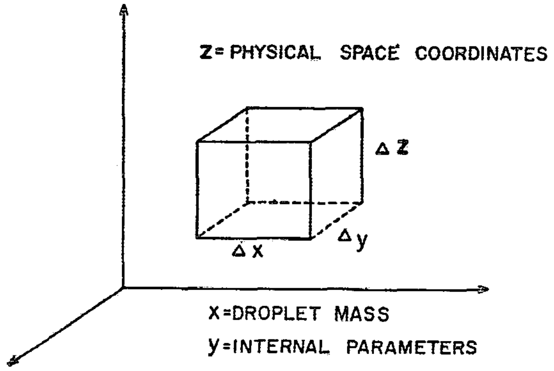

FIG. 1. The orthogonal coordinates of f space.

3. The particle density function

The parameters which describe a particle, and which will be looked upon as orthogonal coordinates in f space, are x = mass, $y = y_i$ = internal parameters, and Z = physical location, with components z_1, z_2, z_3. The particle density function f is then defined such that

$$f(x,y,Z)\Delta x\Delta y\Delta Z = \Delta n(x,y,Z) \quad (1)$$

is the number of particles (or, if less than one, the probability of their existence) in the volume element $\Delta x\Delta y\Delta Z$ of f space at time t (Fig. 1).

Time t is the independent parameter of f, but is not one of the coordinates of f space, and so is not explicitly written in (1). The internal parameter y is here written as one coordinate for convenience. The generalization to more dimensions will be obvious.

The external conditions affecting the growth rate will appear, in the case of condensation, in an auxiliary equation relating $\dot{x}$ to x, y, Z, and the environment. In the case of collection, the environmental effect will enter through the kernel of the integral equations which describe the collection process.

Each particle is represented by a point in f space, i.e., there is a one-to-one correspondence of points in f space and particles in the cloud.

The points in f space are free to move. This motion we call "drifting" (Fig. 2).

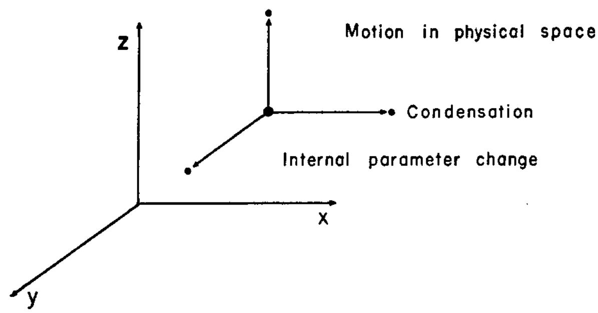

FIG. 2. Motion of a point in f space, called "drifting."

A point moving in the Z direction represents the motion of the corresponding particle in physical space. If the point moves in the x direction it indicates that the particle is gaining mass by condensation of vapor. Motion of the point in the y direction indicates that the internal parameter is changing while mass and position remain constant.

The number of points in a small volume of f space may change with time in three ways, i.e., by drifting, by extinction (that is, just disappearing), and by creation (appearing).

Extinction of a point occurs when its particle has collided and coalesced with another. The point disappears because that particle (that is, the particle of that size and at that Z) ceases to exist. Simultaneously, a new point appears in another position in f space which represents the formation of a new particle having the sums of the masses and the internal parameters of the two coalescing ones. Thus, collection is represented in f space by the extinction of two points and the creation of one.

Our first problem is to write an equation expressing the change in the number of points in a small volume of f space.

4. The growth equation

Under the action of drifting alone, the number of points in f space is conserved. Therefore, the change in f due to drifting is given by the continuity equation with the divergence taken over all dimensions of f space; thus,

$$\left(\frac{\partial f}{\partial t}\right)_{\text{drift}}=-\frac{\partial}{\partial x}(\dot{x}f)-\frac{\partial}{\partial y}(\dot{y}f)-\frac{\partial}{\partial Z}(\dot{Z}f). \quad (2)$$

The change in f due to collection is given by two integral expressions, one accounting for the rate of increase of points and the other accounting for the rate of decrease of points. We shall designate these integrals by I_{gain} and I_{loss}, respectively. The total rate of change of f, then is the sum of that due to drifting and that due to creation and extinction of points, i.e.,

$$\frac{\partial f}{\partial t}=-\frac{\partial}{\partial x}(\dot{x}f)-\frac{\partial}{\partial y}(\dot{y}f)-\frac{\partial}{\partial Z}(\dot{Z}f)+I_{\text{gain}}-I_{\text{loss}}. \quad (3)$$

The total derivatives in x, y and Z (given by the dot over the parameters) must be found from auxiliary equations which include the effect of the environment. Using E to indicate all the effects of the environment of the particles, we write equations for the change in mass, type and position of the particles in the form

$$\left.\begin{aligned}\dot{x}&=\dot{x}(x,y,Z,E)\\ \dot{y}&=\dot{y}(x,y,Z,E)\\ \dot{z}&=\dot{z}(x,y,Z,E)\end{aligned}\right\}. \quad (4)$$

When the collection kernel V of I_{gain} and I_{loss} is given as a function of the same parameters, then the growth equation is completely specified.

5. The environment

Some properties of the environment change due to interaction with the particles. Let us take as an example the vapor pressure.

The "complete" density function over water mass must include the molecular water vapor as well as the water in the droplets. The mass x_m of the molecule is the lower limit. It would be very inconvenient, however, to include molecular sized particles in our droplet density function so we divide the distribution into two size groups by somewhat arbitrarily assigning the boundary x_0. Mass agglomerations $>x_0$ are called droplets; those $<x_0$ are assigned to the vapor pressure (Fig. 3). Even though the interaction between sizes smaller and larger than x_0 is fundamentally one of collection, we call it condensation.

In order to derive an expression for the change in the water vapor mass $q(z)$ we let

$$\begin{aligned}\frac{\partial q}{\partial t}&=\left(\frac{\partial q}{\partial t}\right)_{\text{advection}}+\left(\frac{\partial q}{\partial t}\right)_{\text{condensation}},\\
&=-\frac{\partial}{\partial Z}(\dot{Z}q)-\left[\frac{\partial}{\partial t}\int_{x_0}^{\infty}dx\int_{y_0}^{\infty}dy\, xf\right]_{\text{condensation}},\\
&=-\frac{\partial}{\partial Z}(\dot{Z}q)-\int_{x_0}^{\infty}dx\int_{y_0}^{\infty}dy\, x\left(\frac{\partial f}{\partial t}\right)_{\text{condensation}},\\
&=-\frac{\partial}{\partial Z}(\dot{Z}q)+\int_{x_0}^{\infty}dx\int_{y_0}^{\infty}dy\, x\frac{\partial}{\partial x}(\dot{x}f),\\
&=-\frac{\partial}{\partial Z}(\dot{Z}q)-\int_{x_0}^{\infty}dx\int_{y_0}^{\infty}dy\,(\dot{x}f),\end{aligned} \quad (5)$$

upon integration by parts. Thus, the rate of change in vapor mass is the divergence of its flux minus the rate of condensation on local particles.

In Lagrangian formulation we have

$$\frac{dq}{dt}=-\int_{x_0}^{\infty}dx\int_{y_0}^{\infty}dy\,(\dot{x}f). \quad (6)$$

This is an interface equation. We must also write one for heat exchange between the particles and the environment. The particular interface equations can be written to suit the particular problem so we will not discuss them further.

6. Discussion of the growth equation

The similarity of (3) to the Boltzmann transport equation is striking; yet, there are some important differences. First, f is not a function of velocity, but it is of mass and type of particles. This has important consequences.

For convenience, we let X represent all coordinates in f space. Then (2) may be written as

$$\left(\frac{\partial f}{\partial t}\right)_{\text{drift}}+\frac{\partial}{\partial X}(f\dot{X})=0, \tag{7a}$$

or, equivalently, in terms of the total derivative,

$$\left(\frac{df}{dt}\right)_{\text{drift}}+f\frac{\partial \dot{X}}{\partial X}=0. \tag{7b}$$

In Hamiltonian mechanics the X is composed of matched pairs of position and momentum coordinates, and the application of the canonical equations leads to the vanishing of the last term in (7b). The total derivative of f is then zero, meaning that the density of points in the vicinity of any given point is constant (Liouville's theorem).

But in our case the second term does not in general vanish and so the density of points near a given point may change. There are no canonical equations to guide the trajectories of the points, and the velocities of the points may vary in space in a manner general enough to allow gathering or dispersion of points. So the second difference is that the Liouville theorem does not apply to f space (Fig. 4).

If we include the internal parameters that add to give the new values upon coalescence, then the collection integrals may be written:

$$I_{\text{gain}}(x,y,Z)=\int_{x_0}^{x/2}dx'\int_{y_0}^{y}dy'f(x-x',y-y',Z)V \times(x-x',y-y';x',y';E)f(x',y',Z), \tag{8a}$$

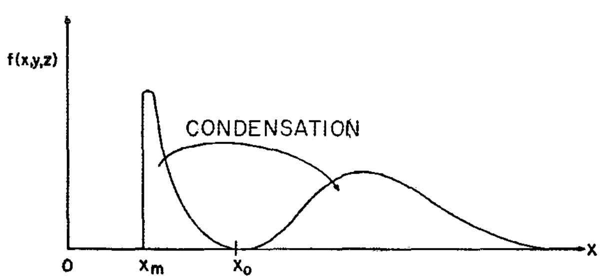

FIG. 3. Division of particle distribution into two size groups. Masses $>x_0$ are called droplets; those $<x_0$ are assigned to the vapor pressure. The interaction is condensation.

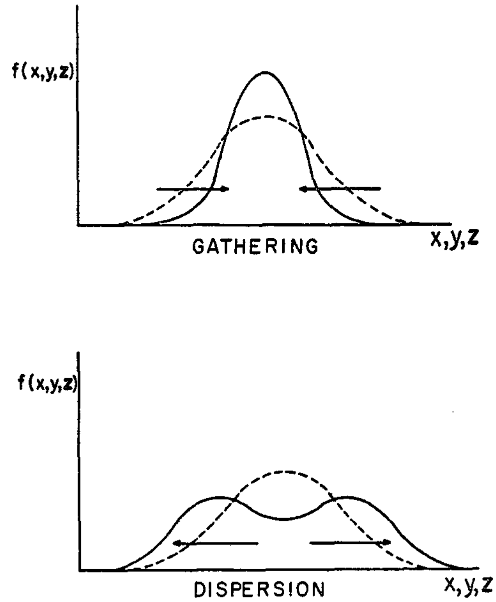

FIG. 4. Gathering and dispersion of points in f space.

$$I_{\text{loss}}(x,y,Z)=\int_{x_0}^{\infty}dx'\int_{y_0}^{\infty}dy'f(x,y,Z)V \times(x,y;x',y';E)f(x',y',Z), \tag{8b}$$

where the primed variables are the parameters of the particle being captured. [A derivation is given in Berry (1965).]

Two intuitive conservation laws can be demonstrated; namely, that under drifting alone the number of points in f space is conserved, and under collection alone the mass of particles is conserved. However, there seem to be no general conservation laws for the combined effects of condensation and collection, and the collection integrals cannot be combined because their limits differ.

7. Summary

A mathematical framework is provided in which the variety of changes in cloud particles can be organized in a cloud model.

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